Journal Of Business & Economics
Vol. 4 No. 1 (Jan-June 2012) pp.30-61

A Circulation Network Model for the Exchange Rate Arbitrage Problem

Carlos Armando de Jesús Cantú García*
Edgar Possani Espinosa**

Abstract

In this article a circulation network model for the detection of arbitrage opportunities in the currencies and securities markets is presented. As an illustration, the study presents its application to the interest rate of the Mexican and American bond market, the interbank loan rate of both countries, as well as to the deposit rates of US and Canada reported in Bloomberg. Deviations of covered interest rate parity imply that there exist a series of transactions that can be carried out to obtain riskless profits by exploiting arbitrage opportunities. The problem of finding arbitrage opportunities is modeled via a generalized maximum flow problem. The maximum flow over the generalized circulation network represent profits from arbitrage, and it is obtained through the application of a minimum cost flow algorithm.

Key Words: Exchange rate arbitrage, financial markets, Circulation Network Model.

1. Introduction

The search for arbitrage is one of the fundamental motors of an economy. Arbitrage consists in performing transactions that generate profits by taking advantage of the difference in prices between two similar assets. In

* Carlos Armando de Jesús Cantú García, University of California, Los Angeles, USA. Carlos Cantú would like to thank the support of CONACYT Reg. No. (364114) and the University of California, Los Angeles.
** Edgar Possani Espinosa, Instituto Tecnológico Autónomo de México, D.F., México. Email at: epossani@itam.mx . Edgar Possani would like to thank the support of Asociación Mexicana de Cultura A.C.

general, there are no opportunities of obtaining profits in the financial markets without incurring some risk. The notion of arbitrage is directly related to the law of one price. This law establishes that in competitive and efficient markets, two assets with similar characteristics and that generate the same cash flow must have the same price (Sercu & Uppal, 1995). Arbitrage, through the law of demand and supply, is the mechanism that ensures the validity of the law of one price.

In this paper, the exchange rate and financial markets are modeled by using circulation networks that are of great interest from the point of view of their applications (Shigeno, 2004). The problem of finding arbitrage opportunities can be formulated by using an exchange rate network, which facilitates the formulation of an appropriate optimization model. In this way, even though the problem can be formulated as a linear programming problem, there exist more efficient minimum-cost flow algorithms that find the solution. These networks are also important because they facilitate visualizing and modeling the relations between a large number of variables.

In order to find arbitrage opportunities in the exchange rate markets, the authors present the problem in a circulation network and solve it as a generalized maximum flow problem. In particular, the problem consist of obtaining the maximum flow that represent the arbitrage profits, along a generalized circulation network representing different transactions that can take place in the exchange and financial markets. Solution of the problem, through optimal flow, indicates the necessary transactions for obtaining profits from arbitrage, or if it is not the case, it indicates there is no arbitrage in that circulation network. To solve this generalized maximum flow problem, the minimum-cost flow algorithm is used as proposed by Goldberg, Plotkin and Tardos (1991). The advantage of using this minimum-cost flow algorithm is that it has lower computational complexity than the traditional algorithms that solve the linear programming problems.

The paper is organized as follows: section 2 present the concepts related to exchange markets, and a discussion about transactions in the financial markets. Section 3 explains how to present transactions in the exchange rate and financial markets by using a network i.e. explain how to present transactions that maximize profits of an arbitrageur as a solution of a linear programming optimization problem. Section 4 explains the concepts related to the generalized maximum flow problem and the optimality conditions for its solution. Section 5 develops the combinatorial minimum-cost flow algorithm proposed by Goldberg, Plotkin and Tardos (1991) to solve the generalized maximum flow problem. In section 6, a data set is used consisting of bid and ask exchange rates and interest rates for Mexico, United States of America (US) and Canada to find the arbitrage opportunities in their exchange rates and financial markets. Finally, section 7 concludes.

2. Exchange Rate and Arbitrage

2.1 The Exchange Rate Markets

An *exchange rate* is the quantity of a currency needed to buy one unit of another currency, or the quantity of a currency received when you sell a unit of another currency. The exchange rate market is composed of two principal segments, the *spot exchange rate* market and the *forward exchange rate* market. The spot market is the foreign exchange market for payments and deliveries of currencies the same day the transaction takes place. The spot exchange rate is denoted as S_t, where t is the current date.

The forward market is the foreign exchange market for payments and deliveries of currencies in a future date. For example, a person might want a bank to quote today the exchange rate of pesos per dollar in three months, and that the transaction takes place in three months at the exchange rate agreed upon today (independent of the effective spot exchange rate at that date). The price agreed upon today for which that person can buy and sell dollars in three months is the forward exchange rate with a three month maturity. In the forward market the exchange rate involved in the transaction is fixed today, that is, there is no uncertainty about the future price. Moreover, the transaction takes place in a future date; this means that there is no exchange of currencies today.

The forward market can be divided in segments depending on the transactions date, and each segment has its own price. The forward exchange rate is denoted as $F_{t,T}$, where t is the current date and T represent the transactions date, which can be measured in weeks, months or years.

2.2 Transactions in the Money Market

The *risk-free market rate of return* in nominal terms is the percentage difference between the initial value (t) and the value at the maturity date (T) of a risk-free asset. The domestic risk-free rate of return is denoted as $r_{t,T}$, where T-t represent the maturity of the contract, and denote the foreign rate of return as $r^*_{t,T}$. Financial institutions express the rate of return on an annual basis (r), so if the maturity of a loan or an investment is less than a year, the effective yield can be calculated as:

$$r_{t,T} = (\text{maturity period in days}/365) \cdot r \tag{1}$$

A *deposit* is a transaction in which a person invests money today and receives money in the future. In order to calculate the amount received, the initial investment is multiplied by $(1+ r_{t,T})$, where $r_{t,T}$ is the effective domestic rate of return at time t for an asset with maturity date T. A *loan* is a transaction in which a person receives money today and pays an equal amount in the future at a cost. For this type of transaction, it is possible to calculate the amount of money received today by multiplying the amount that has to be paid in the future by $(1/(1+ r_{t,T}))$. For the international market, the same transactions can be defined but by using the effective foreign rate of return $r^*_{t,T}$.

2.3 The Law of One Price and the Interest Rate Parity

The law of one price establishes that, in competitive markets, two similar assets must have the same price. Two assets can be considered similar if they have the same maturity date, liquidity and default risk. In particular, two assets are considered similar if *r* generates the same cash flow.

Suppose two assets that generate the same cash flow don't have the same price then the owner of the overvalued asset could simultaneously sell this asset and buy the cheaper one thus obtaining a profit without incurring any additional risk. This type of transaction is called circular arbitrage since it involves buying and selling an asset simultaneously. That is, arbitrage consists in obtaining profits by taking advantage of the price difference between two similar assets.

Circular arbitrage generates an excess supply of the overvalued asset and an excess demand of the undervalued asset; as a consequence the difference in price of the assets will be reduced. If there are perfectly competitive markets, this process will stop until the price of two assets is the same and there are no profits from arbitrage. If there are transaction costs, the purchase and sale of assets will stop when the cost of buying the undervalued asset and selling the overvalued asset exceeds the difference between their prices.

Although this mechanism result in the equalization of prices for assets that generate the same cash flow, in practice there are three reasons why profits can be obtained by using arbitrage:

1) Markets can be inefficient in the sense that investors don't react immediately, or are not aware of the profits that can be made in the market.

2) In periods with expectations of high volatility in the foreign exchange market, speculative pressures can result in an increase of the supply of funds that generate arbitrage in the forward market. That is, in periods with high financial uncertainty and instability, the same currency can be quoted at different values in the forward market depending on the perception that each financial institution has about the future value of the exchange rate.

3) The difference in price of assets can be the result of the default risk

implicit in the contracts.

In the foreign exchange and money markets, the arbitrageur examines the current quotes for the exchange rates and interest rates, in order to look for the possibility of obtaining riskless instantaneous profits by means of an appropriate combination of transactions. If the markets are perfectly competitive, every participant in the market will be aware of these profits and will try to make the same transactions. These massive transactions will have an effect that some of the prices of the assets involved in the trades will adjust until they reach their equilibrium price. The equation that relates these equilibrium prices is the following:

$$F_{t,T} = S_t \frac{1 + r_{t,T}}{1 + r^*_{t,T}} \tag{2}$$

This equation shows that, in equilibrium, the forward exchange rate must be equal to the spot exchange rate adjusted by a factor that depends on the ratio between the domestic interest rate and the foreign interest rate. Solving for the domestic interest rate, the following equation is obtained:

$$1 + r_{t,T} = \frac{1}{S_t} \cdot \left(1 + r^*_{t,T}\right) F_{t,T} \tag{3}$$

The right hand side of equation 3 represents the profits obtained from a covered foreign investment. That is, profits generated by changing pesos for dollars in the spot market, investing dollars in the US, and hedging the foreign exchange risk by selling dollars in the forward market. This equation shows that the profits obtained from a covered foreign investment have to be equal to the cost of a loan in the domestic market. For this reason, this equation is known as the covered interest rate parity.

However, the most common equation for the interest rate parity (3) does not take into account transaction costs. These costs can be expressed as the difference between the bid and ask prices of exchange rates and interest rates; that represent inventory, information and processing costs related to the transactions of an asset. An investor pays the ask price when he solicits a loan and receives the bid price when he makes a deposit. In the same way, an investor pays the ask price of the exchange rate when he buys foreign currency and receives the bid price when he sells foreign currency. Taking into account the difference between the bid and ask prices of exchange and interest rates, the conditions that ensure that there are no profits from circular arbitrage are:

$$1 + r_d \geq \frac{1}{S_d} \cdot \left(1 + r_o^*\right) \cdot F_o \tag{4}$$

$$1 + r_d^* \geq S_o \cdot \left(1 + r_o\right) \cdot \frac{1}{F_d} \tag{5}$$

where the *d* subscript refers to the ask prices and the *o* subscript refers to the bid prices. Moreover, the maturity date for the contracts in the forward exchange market and the loan and deposit transactions are the same. The next section explains how to represent the transactions introduced in this section using network diagrams.

3. Network Model for the Foreign Exchange Arbitrage Problem

3.1 Network Representation of the Foreign Exchange and Money Markets

Throughout this paper, it is assumed that Mexico is the domestic country (Q_t) and the US is the foreign country (Q^*_t). Moreover, the maturity date for deposits, loans and contracts in the forward exchange market is assumed to be of three months. Transactions in the spot market can be represented as a flow from an initial or head node (the initial position or the quantity of a currency given to a bank) to a terminal or tail node (the final position after the transaction or the quantity received of the currency from the bank). The direction of a transaction is represented by an arrow or arc and the gain associated with that arc (the exchange rate or its inverse) is shown beside the arrow. Transactions in the forward exchange market can be represented in the same way. The selling of a foreign currency in the forward exchange market implies that today (t) a person undertakes to pay an amount of foreign currency in the future (T) and in return, he receives an amount of the domestic currency in the future. In this case, the initial position is an amount Q_T, the final position is an amount Q^*_T, and the gain is the inverse of the forward exchange rate with maturity date T.

For a deposit, the transaction can be represented as a flow from an initial node Q_t (the amount of money lent today) to a terminal node Q_T (the amount of money received in the future), and the gain corresponds to the effective return rate. For a loan, the arc stems from the quantity that has to be paid in the future Q_T to the quantity received today Q_t, and the gain corresponds to the inverse of the loan rate. The network that combines the four types of transactions mentioned before is shown in Figure 1[1].

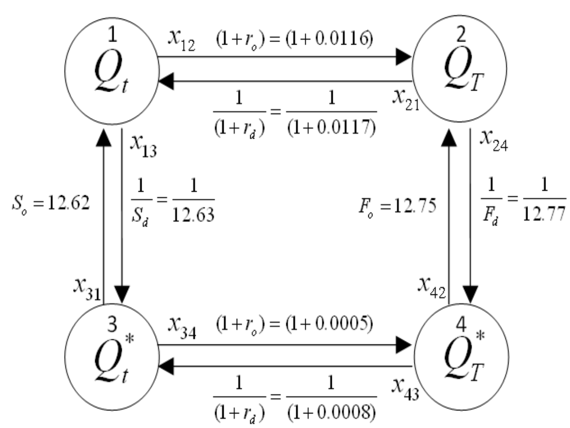

Fig. 1

Using this diagram, two types of transactions involved in circular arbitrage are presented taking into account the difference between the bid and ask price in the money markets.

[1] The data was obtained from Bloomberg and corresponds to March 15, 2010. The interest rate for Mexico corresponds to the three- month CETES rate and for the US corresponds to the three-month deposit rate.

Arcs that move counterclockwise correspond to the transactions in equation (4), and arcs that move clockwise correspond to the transactions in equation (5). The value of the flow changes between currencies when it moves vertically in the diagram. The value of the flow is increased or reduced when it moves horizontally in the diagram via the interest rate. Each node is numbered and each arc has a variable assigned to it, which represents the value of the flow in that arc. The first subscript in the arc's variable corresponds to the node it comes from and the second subscript corresponds to the node it goes to.

3.2 Linear Programming Formulation

The solution to the linear programming problem represents the flow of optimal transactions in the arcs and the objective function evaluated at the solution represents the riskless gains from arbitrage. The linear programming problem that corresponds to the exchange rate network can be formulated using three rules:

1) The objective is to maximize the flow out of the node representing the period and currency desired.

2) For each node, a constraint is added that represents the flow conservation condition. That is, all the flow that enters a node must also come out. Flow into a node is considered positive and flow outward is considered negative.

3) For each arc, a constraint is added that represent the upper bound for the credit and trading limits.[2]

In this paper, it is assumed that the function to maximize is the gain in pesos in three months. In order to specify this objective function, an arc is

[2] The credit limits represent the maximum quantity that an investor can lend or borrow without changing the interest rate. The trading limits represent the maximum amount a trader can buy or sell of currencies without affecting the interest rate. These limits are also associated with the maximum quantities that can be traded immediately.

added that leaves node 2 (x_2), which is the value to maximize. Moreover, to each arc, its corresponding upper bound is added. The complete diagram is represented in Figure 2:

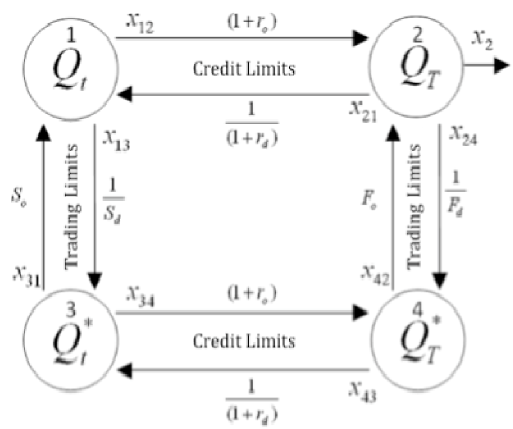

Fig. 2

The linear programming problem that corresponds to the network diagram of Figure 2 is the following:

$$
\begin{aligned}
&\max\ x_2 \\
&s.t. \\
&S_o x_{31} + \left(\frac{1}{1+r_o}\right) x_{21} - x_{13} - x_{12} = 0 \\
&\frac{1}{S_d} x_{13} + \left(\frac{1}{1+r_o^*}\right) x_{43} - x_{31} - x_{34} = 0 \\
&F_o x_{42} + (1+r_d) x_{12} - x_{24} - x_{21} = x_2 \\
&\frac{1}{F_d} x_{24} + (1+r_d^*) x_{34} - x_{42} - x_{43} = 0
\end{aligned}
$$

trading limit of pesos in the spot market $\geq x_{13}$
trading limit of dollars in the spot market $\geq x_{31}$
trading limit of pesos in the forward market $\geq x_{24}$
trading limit of dollars in the forward market $\geq x_{42}$
credit limit in México $\geq x_{12}, x_{21}$
credit limit in the US $\geq x_{34}, x_{43}$
$x_i \geq 0\ \forall i$

Where the first four constraints correspond to the flow conservation conditions for nodes 1, 3, 2 and 4 respectively, the rest of the constraints correspond to the arcs' credit and trading limits. If the objective function evaluated at the solution of the problem equals zero, it implies that for this network there are no gains from circular arbitrage.

There might be more opportunities to obtain gains from arbitrage when more currencies and periods are considered simultaneously. An arbitrageur could benefit by borrowing in several currencies and depositing the funds in others. Exchange rate and interest rate quotes change continually, and they could be used directly in the network model to obtain the optimal transactions using the solution to the linear programming problem. However, network flow algorithms that have a lower computational complexity than solution methods for linear programming have been developed.

4. Generalized Maximum Flow Problem

4.1 Definitions

A *circulation network* is a directed graph $G=(N,A,u,s)$, where N is the set of nodes and A is the set of arcs. Each arc $a \in A$ can be represented as a pair of order nodes $a=(i, j)$ such that $i, j \in N$, where the order of the nodes represent the direction of the arc. It is assumed that the cardinality of the set of nodes is n and the cardinality of a set of arcs is m. Each arc has an associated capacity, which represents the maximum amount of flow that can go through the arc. This capacity can be represented by a function $u{:}A \rightarrow \mathbb{R}^+$. The graph G has a special node s called the *source*, which represents the final destination of the flow.

A *path* is a sequence of arcs $P=(a_1,\ldots,a_{q-1})$ with $a_1=(i_1,i_2)$, $a_2=(i_2,i_3),\ldots,a_{q-1}=(i_{q-1},i_q)$, where $q-1$ is the length of the path. Node i_1 is the initial node of the path, and node i_q is the final node. An *elementary path* is a path that does not contain a node more than one time. The set of all elementary paths that start in node i_1 and end in node i_q is denoted by $P_{1,q}$.

A *cycle* is a path of length greater than zero such that the initial node and the final node is the same. A cycle can be presented as $C=(a_1,\dots,a_{q-1})$ with $a_1=(i_1,i_2)$, $a_2=(i_2,i_3),\dots,a_{q-1}=(i_{q-1},i_q)$, and $i_1=i_q$. An *elementary cycle* is a cycle that does not contain the same node more than once, with the exception of the node where the cycle begins.

In the *generalized flow problem,* each arc has a *gain* associated to it. This gain is given by the function $\gamma:A\rightarrow\mathbb{R}^{+}\setminus\{0\}$. In the case of ordinary flows, if k units exit node i, then k units arrive to node j. This means that the gain is always one in every arc of the graph. In the case of generalized flows, if k units of flow exit node i, then $k*\gamma(i,j)$ units arrive to node j. In this case, not all gains are necessarily equal to one.

If $\gamma(i,j)>1$, then (i,j) is a gain arc, and if $\gamma(i,j)<1$, then (i,j) is a loss arc. The gain of a path or cycle is given by the product of the gains of the arcs that make up the path or cycle, i.e. $\prod_{\{(i,j)\in P\}}\gamma(i,j)$.

A cycle with a unit gain is called a *unitary cycle*, a cycle with a gain greater than one is called a *flow-generating cycle,* and a cycle with a gain less than one is called a *flow-absorbing cycle.* It is assumed that for each arc (i,j), there exist an arc (j,i) where the following condition is satisfied:

$$\gamma(i,j)=\frac{1}{\gamma(j,i)} \tag{6}$$

This condition is known as the *gain antisymmetry constraint.* These inverse arcs can be introduced without loss of generality because, if these arcs are not included in the original graph, they can be added and their capacity is set to zero.

A *generalized pseudoflow* is a function $f: A\rightarrow\mathbb{R}^{+}$ that satisfies the following conditions:

Capacity Constraint:

$$f(i,j) \le u(i,j) \qquad \forall (i,j) \in A \tag{7}$$

Gain Antisymmetry Constraint:

$$f(i,j) = -\gamma(j,i) * f(j,i) \qquad \forall (i,j) \in A \tag{8}$$

Given a generalized pseudoflow f, the residual capacity function $u_f{:}A \to \mathbb{R}^+$ is defined as $u_f(i,j)=u(i,j)-f(i,j)$. The residual graph with respect to f is given by $G_f=(N,A_f)$, where $A_f=\{(i,j)\in A|\ u_f(i,j)>0\}$. The *excess* in a node i with respect to the generalized pseudoflow f is defined as:

$$e_f(i) = -\sum_{(i,j)\in A|f(i,j)>0} f(i,j) + \sum_{(j,i)\in A|f(j,i)>0} \gamma(i,j)\cdot f(i,j) \tag{9}$$

which represent the net flow into a node i. The definition of an excess can be simplified by using the gain antisymmetry constraint:

$$\begin{aligned} e_f(i) &= -\sum_{(i,j)\in A|f(i,j)>0} f(i,j) + \sum_{(j,i)\in A|f(j,i)>0} \gamma(i,j)\cdot f(i,j) \\ &= -\sum_{(i,j)\in A|f(i,j)>0} f(i,j) + \sum_{(j,i)\in A|f(j,i)<0} -f(i,j) \\ &= -\sum_{(i,j)\in A} f(i,j) \end{aligned}$$

A node i has an *excess* if $e_f(i)>0$ and has a *deficit* if $e_f(i)<0$. The value of a pseudoflow f is given by the excess at the source $e_f(s)$.

A *generalized flow* is a function $g{:}\ A \to \mathbb{R}^+$ such that, in addition to satisfying the capacity and antisymmetry constraints, it satisfies the following flow conservation constraint:

$$e_g(i)=0 \qquad \forall i\in N\backslash\{s\} \tag{10}$$

The *generalized circulation problem* can be represented by the tuple (G=(N,A),u,γ,s) where G is a circulation network, u is a capacity function, γ is a gain function, and s is the source. The generalized maximum flow problem consists in finding the generalized flow g^* that maximizes its value:

$$\begin{aligned}
&\max \quad e_g(s) \\
&s.t. \\
&e_g(i) = 0 \qquad && \forall (i,j) \in N \setminus \{s\} \\
&g(i,j) \le u(i,j) && \forall (i,j) \in A \\
&g(i,j) = -\gamma(j,i)\cdot g(j,i) && \forall (i,j) \in A
\end{aligned}$$

A *generalized augmenting path* (GAP) is a flow-generating cycle and a path from one of the nodes of the cycle to the source. This path can be trivial if the source is included in the cycle. Given a generalized flow and a GAP in the residual graph, the flow can be augmented along the GAP, therefore increases the value of the current circulation. The following theorem shows the optimality condition for the generalized maximum flow problem[3]:

Theorem 1 *A generalized flow is optimal if and only if the residual graph associated with the flow does not contain any generalized augmenting paths.*

4.2 Restricted Problem

The circulation problem is *restricted* if in the residual graph of the pseudoflow f_0, defined by $f_0(i,j)=0 \ \ \forall(i,j)\in A$, every flow generating cycle pass through the source. Using this definition, the optimality condition for a generalized flow can be written as[4]:

[3] Onaga, K., 1967, Optimal Flows in General Comunication Networks, Journal of the Franklin Institute, 283, 308-327.
[4] Onaga, K., 1966, Dynamic Programming of Optimum Flows in Lossy Communication Nets, IEEE Transactions on Circuit Theory, 13, 282-287.

Theorem 2 *Given a restricted problem, a generalized flow is optimal if and only if the residual graph associated with the flow does not contain any flow generating cycles.*

The advantage of transforming the problem to its restricted form is that, depending on the excess at the nodes of a pseudoflow *f*, the existence of directed paths in the residual graph from every node to the source or from the source to every node can be ensured. This result is formalized in the following lemma[5]:

Lemma 1 *Let f be a generalized pseudoflow in a restricted network, then:*

1) If the excess at every node other than at the source is nonnegative, then for every node i there exist a path from node i to the sink in the residual graph of the pseudoflow f.

2) *If the residual graph of the pseudoflow f has no flow-generating cycles and the excess at every node other than at the source is nonpositive, then for every node i there exist a path from the source to node i in the residual graph of the pseudoflow f.*

4.3 Node Labeling

The process of changing the units of measure of a problem in a circulation network is called *relabeling*, and the equivalent problem obtained after relabeling is called the *relabeled* problem.

A *label* is a function $\mu:N\rightarrow\mathbb{R}^+$, where $\mu(i)$ is the label of *i,* and denote the number of previous units for each new unit in *i.* Given a generalized circulation problem $(G=(N,A),u,\gamma,s)$ and a function $\mu:N\rightarrow\mathbb{R}^+$, the relabeled problem is given by $G_\mu=((N,A),u_\mu,\gamma_\mu,s)$ where:

[5] Goldberg, A. V., Plotkin, S. A. y Tardos, É., 1991, Combinatorial Algorithms for the Generalized Circulation Problem, Mathematics of Operations Research, 16, 351-381.

$$u_{\mu}(i,j)=\frac{u(i,j)}{\mu(i)},\quad \gamma_{\mu}(i,j)=\frac{\gamma(i,j)\mu(i)}{\mu(j)} \tag{11}$$

Given a generalized pseudoflow f and a label μ, the *relabeled residual capacity* and the *relabeled excess* are defined as:

$$u_{f,\mu}(i,j)=\frac{u(i,j)-f(i,j)}{\mu(i)},\quad e_{f,\mu}(i)=\frac{e_f(i)}{\mu(i)} \tag{12}$$

If f is a generalized pseudoflow in a circulation problem (G=(N,A),u,γ,s), then $f_\mu(i,j)$=((f(i,j))/($\mu(i)$)) is a generalized pseudoflow in the relabeled problem G_μ=((N,A),u_μ,γ_μ,s). Moreover, the residual graphs of f and f_μ are the same.

Two symmetric methods of relabeling are the *canonical relabeling from the source* and the *canonical relabeling to the source*. The first method is used when one wants to send additional flow from the source, and the second when one wants to send additional flow to the source.

The method of canonical relabeling from the source can be used when every node $i \in N$ can be reached from the source by a path in the residual graph of the generalized pseudoflow f.

Define $\mu(s)$=1 and for each node $i \in N \backslash \{s\}$, the canonic label $\mu(i)$ is defined as the highest gain from all the elementary paths from s to i in the residual graph, that is:

$$\mu(i)=\max_{P_{s,i}}\left\{\prod_{(j,k)\in P_{s,i}}\gamma(j,k)\right\} \tag{13}$$

The canonic label corresponds to the amount of flow that can reach node i, if one unit of flow is sent through the most efficient elementary path from s to i in the residual graph, ignoring the capacity constraints along the path.

The canonical relabeling to the source is defined in a similar way and it can be used when the source can be reached from every node $i \in N$. Define $\mu(s)=1$ and for each node $i \in N \setminus \{s\}$, the canonic label $\mu(i)$ is defined as the inverse of the highest gain from all the elementary paths from i to s in the residual graph, that is:

$$\frac{1}{\mu(i)} = \max_{P_{i,s}} \left\{ \prod_{(j,k) \in P_{i,s}} \gamma(j,k) \right\} \tag{14}$$

The following properties are useful in determining the optimality of a generalized flow[6]:

Theorem 3 *After a canonic relabeling from the source;*

1) Every arc (i,j) with a non-zero residual capacity, other than the arcs entering the source, has $\gamma_\mu(i,j) \leq 1$.

2) For every node i, there exist a path from s to i in the residual graph with $\gamma_\mu(i,j)=1$ *for all arcs on the path.*

3) The most efficient flow-generating cycles consist of paths $P_{s,i}$, *for some* $i \in N$, *with* $\gamma(j,k)=1$ *along the path, and the arc* $(s,i) \in A_f$ *such that* $\gamma_\mu(s,i)=max(\gamma_\mu(i,j) \mid (i,j) \in A_f)$.

Theorem 4 *A generalized flow g in a restricted problem is optimal if and only if there exist a label* μ *such that* $\gamma_\mu(i,j) \leq 1$ *for every arc in the residual graph of the generalized flow.*

A label μ is optimal if there exist a generalized flow g such that μ and g satisfy the conditions of Theorem 4. Canonic relabeling helps to simplify

[6] Goldberg, A. V., Plotkin, S. A. y Tardos, É., 1991, Combinatorial Algorithms for the Generalized Circulation Problem.

the optimality conditions of a generalized flow, and is a useful method to find the most efficient flow-generating cycles. In the next section, the basic definitions of the minimum-cost flow and the interpretation of a minimum-cost flow as a generalized pseudoflow in the generalized maximum flow problem are presented.

4.4 Minimum-Cost Flow Problem

The concepts of the minimum-cost flow problem are similar to the concepts of the generalized maximum flow problem. The most important difference between these two problems is that, in the generalized maximum flow problem, the flow is transformed by a factor proportional to the gain associated to the arc when it goes from one node to another. On the other hand, in the minimum-cost flow problem, the flow stays the same as it goes from one node to another. Moreover, in the minimum-cost flow problem, each node has a demand or supply that has to be satisfied, and each arc has an associated cost.

Let $G= (N,A,u,s)$ be a circulation network and $u:A\rightarrow\mathbb{R}^+$ a capacity function. A *pseudoflow* is a function $f_c:A\rightarrow\mathbb{R}^+$ that satisfies the following conditions:

Capacity Constraint:

$$f_c(i,j) \leq u(i,j) \qquad \forall (i,j) \in A \qquad (15)$$

Antisymmetry Constraint:

$$f_c(i,j) = -f_c(j,i) \qquad \forall (i,j) \in A \qquad (16)$$

Given a pseudoflow f_c, define the *residual capacity* function $u_{fc}:A\rightarrow\mathbb{R}^+$ as $u_{fc}(i,j)=u(i,j)-f_c(i,j)$. The *residual graph* with respect to f_c is given by $G_{fc}=(N,A_{fc})$, where:

$$A_{fc}=\{(i,j)\in A|u_{fc}(i,j)>0\} \tag{17}$$

A *cost function* is a function defined on the arcs $c: A\to\mathbb{R}$. This function represents the cost of each unit of flow that goes through an arc. Without loss of generality, the costs are assumed as antisymmetyric, i.e. $c(i,j)$=-$c(j,i)$ $\forall(i,j)\in A$. The *cost* of a pseudoflow is defined as:

$$c(f_c)=\sum_{(i,j)\in A|f(i,j)>0} f(i,j)\cdot c(i,j) \tag{18}$$

A *supply/demand function* is a function defined on the nodes, $b: N\to\mathbb{R}$. This function represents the supply or demand of flow on each node. If $b(i)>0$ then the node is a supply node, and if $b(i)<0$ then the node is a demand node.

A *flow* in the minimum-cost flow problem is a function g_c: $A\to\mathbb{R}^+$ such that, in addition to the capacity and antisymmetry constraints, it satisfies the following condition:

$$\sum_{(i,j)\in A} g_c(i,j)=b(i)\forall i\in N \tag{19}$$

The minimum-cost flow problem can be represented by the tuple (G= (N, A), u, c,b) where G is a circulation network, u is a capacity function, c is a cost function, and b is a supply/demand function. The minimum-cost flow problem consists in finding the flow g^*_c that minimizes its cost:

$$\begin{aligned}
&\max \quad c(g_c) \\
&s.t. \\
&\sum_{(i,j)\in A} g_c(i,j) = b(i) && \forall (i,j)\in N\setminus\{s\} \\
&g_c(i,j)\le u(i,j) && \forall (i,j)\in A \\
&g_c(i,j) = -g(j,i) && \forall (i,j)\in A
\end{aligned}$$

The theorem that describes the optimality conditions for a flow is the following[7]:

Theorem 5 *A flow is optimal if and only if the residual graph associated with that flow does not contain any cycle with a negative cost.*

The *interpretation* of a pseudoflow f_c is a generalized pseudoflow f, such that:

$$f(i,j) = \begin{cases} f_c(i,j) & si\ f_c(i,j) \geq 0 \\ -\gamma_\mu(j,i) f_c(j,i) & e.o.c \end{cases}$$

The following lemma relates a pseudoflow f_c with its interpretation[8]:

Lemma 2 *The residual graph of a pseudoflow f_c and the residual graph of its interpretation are the same.*

5. Minimum-Cost Flow Algorithm

In this section, the Minimum-Cost Flow Algorithm (MCF) is presented that solves the generalized maximum flow problem. This algorithm was developed by Goldberg, Plotkin & Tardos (1991). The algorithm is called as minimum-cost flow because it is based on a minimum-cost flow subroutine. This algorithm was chosen to solve the network problem of foreign exchange arbitrage because it is the first combinatorial algorithm that has a polynomial computational complexity. At each iteration, the algorithm solves a more simple flow problem and interprets the result as an increase in the flow of the generalized circulation network. The algorithm assigns a cost $c(i,j)=-\log(\gamma(i,j))$ to each arc and solves the corresponding minimum-cost flow problem.

---

[7] Ahuja, R. K., Magnanti, T. L., & Orlin, J., 1993, Network Flows: Theory, Algorithms, and Applications, Prentice Hall.

[8] Goldberg, A. V., Plotkin, S. A. & Tardos, É., 1991, Combinatorial Algorithms for the Generalized Circulation Problem.

5.1 Algorithm

The first step of the algorithm is to transform the problem into its restricted form and start with the pseudoflow g_0, defined by $g_0(i,j)=0$, $\forall(i,j)\in A$. During the first iteration, the algorithm takes advantage of the fact that all flow-generating cycles pass through the source in order to generate a positive excess at the source. Subsequent iterations use this excess to balance out the deficits in the rest of the nodes of the graph by relabeling canonically the residual graph. Then, the algorithm solves the corresponding minimum-cost flow problem that satisfies the deficits generated in the previous iteration and interprets the result as a generalized pseudoflow. The algorithm stops when the canonical labels of the residual graph corresponding to the next iteration satisfy that $\gamma_\mu(i,j)\leq 1\ \forall(i,j)$, and there is no excess in any node except at the source. The steps of the algorithm are detailed below:

MCF Algorithm

1) Transform the generalized circulation problem into its restricted form and start with the generalized flow g, defined by $g(i,j)=0$ $\forall(i,j)\in A$.

2) Find the canonical labels from the source μ. If $\gamma_\mu(i,j)\leq 1$ for every arc in the residual graph and $\forall i\in N\backslash\{s\}$ $e_{\mu g}(i)=0$, then stop the algorithm. The current generalized flow is optimal. Otherwise, go to step 3.

3) Introduce the costs $c(i,j)=-\log(\gamma_\mu(i,j))$ on every arc in the graph.

4) Find the minimum-cost pseudoflow f_c' of the relabeled residual graph with capacity μ. The supply in every node equals $b(i)=-e_{\mu g}(i)$ $\forall i\in N\backslash\{s\}$.

5) Let g' be the interpretation of f_c'. Update the solution as

$g(i,j)=g(i,j)+g'(i,j)\mu(i) \ \forall (i,j)\in A$. Go back to step 2.

In the next section, some results are shown that provide a better understanding of the algorithm and present its computational complexity. Afterwards, an example of the implementation of the algorithm is shown.

5.2 Analysis of the MCF Algorithm

The most relevant property of an optimal minimum-cost flow, for its application in the generalized maximum flow problem, is that the residual graph of the optimum flow has no negative cost cycles (theorem 5). This property together with lemma 2 gives the following result[9]:

Corollary 1 *The residual graphs of the generalized pseudoflows generated in step 5 of the algorithm have no flow-generating cycles..*

A problem that could arise when implementing the algorithm is not being able to use the canonical relabeling from the source. However, using the result from the previous corollary, and taking into account that the problem is in its restricted form, then, by lemma 1, for every node *i* there exist a path from the source to node *i* in the residual graph of the pseudoflows generated by the algorithm. Therefore, at every iteration, the canonical relabeling from the source can always be used. This result and other two results are presented below[10]:

Lemma 3 *The following results are true for a generalized pseudoflow g generated by the MCF algorithm:*

1) *The canonical relabeling from the source can always be used in the residual graph of g.*

2) *All excesses, except at the source, are non-positive.*

[9] Goldberg, A. V., Plotkin, S. A. y Tardos, É., 1991, Combinatorial Algorithms for the Generalized Circulation Problem.
[10] Ibid.

3) *For the relabeled problem, there exist a minimum-cost pseudoflow f_c such that:*

$$\sum_{(i,j)\in A} f_c(i,j) = -e_{g,\mu}(i) \quad \forall i \in N \backslash \{s\}$$

Consider a generalized pseudoflow *f* at the beginning of some iteration of the MCF algorithm. The fact that there are only deficits in the nodes of the residual graph of *f*, except at the source, and that there are no flow generating cycles, implies that the excess at the source is an overestimate of the maximum possible excess. The algorithm stops when it finds a generalized flow *g* for which the excess at every node, except at the source, is zero, and every relabeled gain of the arcs in the residual graph is less than one. Therefore, by theorem 4, the generalized flow obtained from the algorithm is optimal. This result is formalized below[11]:

Theorem 6 *Every iteration of the algorithm can be implemented in polynomial time and the generalized flow g^* obtained from the algorithm is optimal.*

To obtain a limit on the running time of the MFC algorithm, it is necessary to decide which minimum-cost algorithm is used as a subroutine. In the work of Golberg, Plotkin and Tardos (1991), they find that the algorithm that provides the best running time is the polynomial algorithm by Orlin (1993). The following theorem presents the minimum running time for the MCF algorithm[12].

Theorem 7 *The MCF algorithm can be implemented to use at most $O(n^2m(m+nlogn)lognlogB)$ arithmetic operations on numbers whose size is bounded by $O(mlogB)$, where n is the number of nodes, m is the number of arcs and B is the maximum of all the gains and capacities associated to the*

[11] Ibid.
[12] Ibid.

arcs.

6. Programming, Data and Implementation

6.1 Matlab Programming

The MCF algorithm was programmed in Matlab using a sequence of subroutines that correspond to the different transformations of the original generalized maximum flow problem.

Once an optimum minimum-cost flow is obtained, the result is interpreted as a generalized pseudoflow and the solution to the generalized maximum flow problem is updated. The algorithm repeats the sequence of subroutines until it finds a generalized flow that satisfies the conditions of theorem 4.

6.2 Data on Foreign Exchange Rates and Interest Rates

Data on spot and forward exchange rates with a three-month maturity for the Mexican peso, the American dollar and the Canadian dollar was obtained from Bloomberg. The database covers every working day from November 24, 2008 to May 11, 2010. These dates were chosen because it was a period of high volatility in the exchange rates and interest rates as a result of the 2007-09 crisis. For each foreign currency, bid and ask prices were obtained.

For the interest rates, the arbitrage problem requires that the interest rates have to be over assets with similar characteristics (liquidity, risk, etc.). Data on three different interest rates was obtained: the bid and ask rate for the three-month government bonds of Mexico and the US; the three-month interbank interest rate for Mexico and the US; and the three-month deposit rate of the US and Canada. The interest rates were obtained from Bloomberg. These rates cover the same period as the foreign exchange rate quotes.

Using this database, three exercises were carried out to determine whether there were opportunities of obtaining riskless profits from circular arbitrage during the period corresponding to the data. The first exercise takes into account the interest rates on government bonds of Mexico and the US. The second uses the interbank interest rates of Mexico and the US, and the last one, the deposit rates of Canada and the US.

6.3 Implementation and Results

The model presented in section 3 (figure 2) was used to implement the MCF algorithm in order to find arbitrage profits in an exchange rate network. However, the generalized maximum flow problem formulation requires that if $\gamma(i,j)$ is the gain of arc (i,j), then the gain of arc (j,i) has to be $(1/(\gamma(i,j)))$. This is not true when there is a difference between the bid and ask price of the exchange rates and interest rates. For example, if dollars were exchanged for pesos at a rate of S_o, the transaction of buying dollars would not be done at an exchange rate of $(1/(S_o))$, but at a rate of $(1/(S_d))$. To solve this problem, for each node, another node is added to represent the difference between the bid and ask transactions. Each pair of bid and ask nodes represent a currency for a given period. In this way, the return arcs can be added, the inverse gain of the original arc can be assigned to them and their capacity can be set to zero. Moreover, between each bid and ask node, an arc with a unitary gain is added and with a large enough capacity to allow the free flow between these nodes. Once the corresponding arcs and nodes are introduced, the circulation

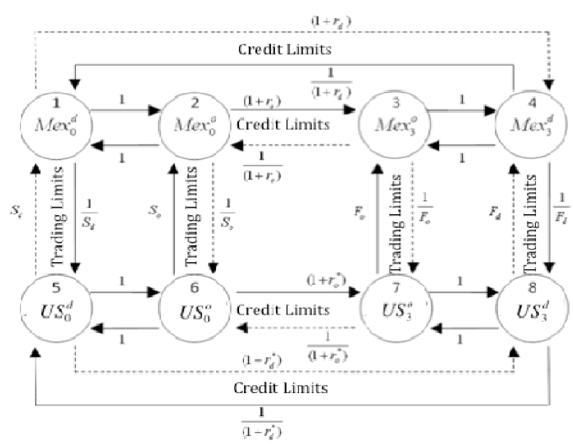

Fig. 3

network can be introduced into the MCF algorithm to calculate the circular arbitrage opportunities. An example of a circulation network with augmented nodes and arcs is shown in figure 3.

For the 297 days considered using the interest rates of government bonds of Mexico and the US, there were 136 days when there were opportunities of obtaining profits from circular arbitrage. The longest period in which there was arbitrage was the period between the 24th of November and the 8th of December of 2008 (18 days). Figure 4 show the interest rates corresponding to this period and the four longest periods where arbitrage was observed are circled.

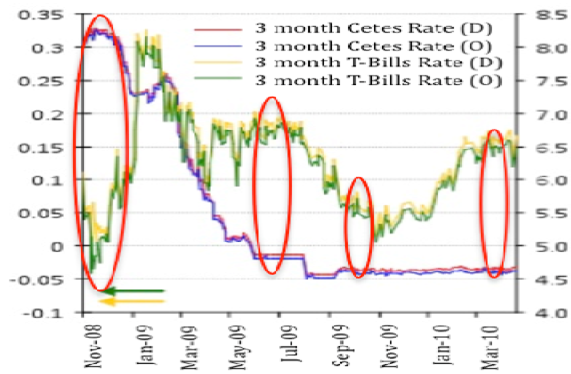

Fig. 4

The second exercise uses the interbank rate for Mexico and the US. In this case, the bid and ask rate is the same. For the 291 days considered, there were opportunities of circular arbitrage in 207 days. The longest period where circular arbitrage was observed was from the 26th of May to the 30th of July of 2009 (46 days). In both cases, using the interbank rate or using the bonds rate, the transactions involved in the periods with the longest arbitrage opportunities consisted in borrowing in Mexico and investing in the US. Moreover, in the longest period where arbitrage was observed using the bonds rate, there were also arbitrage opportunities using the interbank rate.

Figure 5 graphs the relations of the interest rate parity of equations (4) and (5) with data of the bonds rate. The relations were normalized such that a value greater than one represent the existence of arbitrage opportunities. A value less than one does not represent arbitrage opportunities, it represents that there would be losses when performing these transactions. Moreover, the graph allows seeing the extent of the deviations from the interest rate parity. The bigger this deviation, the higher the profits are from arbitrage.

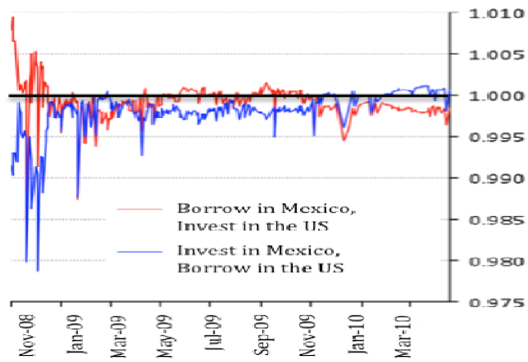

Fig. 5

One of the reasons why more arbitrage opportunities were found in the second exercise is that the bid and ask rates considered were the same. In this way, the transaction costs implicit in the difference between the bid and ask price was eliminated and it was feasible to find more arbitrage profits. Even though the first exercise considers the difference between the bid and ask rates, this difference represents a lower bound of the transaction costs involved in these transactions. Moreover, when analyzing the interest rates, another factor that has to be taken into consideration is the liquidity of the loan and deposit contracts. Even if the implicit risk of the domestic and foreign contracts is the same, the difference in the liquidity of the assets would reduce the opportunities of obtaining profits from arbitrage. This leads to the conclusion that, although arbitrage opportunities were found in a little less than half the days in the first exercise and a little more than two thirds of the days for the second exercise, these results must be interpreted as an upper bound on the arbitrage opportunities that could be carried out in practice.

For the third exercise, the deposit rates as reported by Bloomberg for the US and Canada were used. In this exercise Mexico was not considered because a similar interest rate could not be found. Also, as the authors wanted to analyze two economies with a similar country risk and where the liquidity of the deposit contracts was the same.

The results show that during the 297 days considered, there were arbitrage opportunities in only 16 of them. The longest period with arbitrage was from the 22nd to the 24th of December of 2008. The transactions corresponding to this period involved borrowing in the US and investing in Canada. Figure 6 shows the interest rate parity deviations using the data from this exercise. It can be noted that the deviations and their magnitude are smaller than in the first exercise.

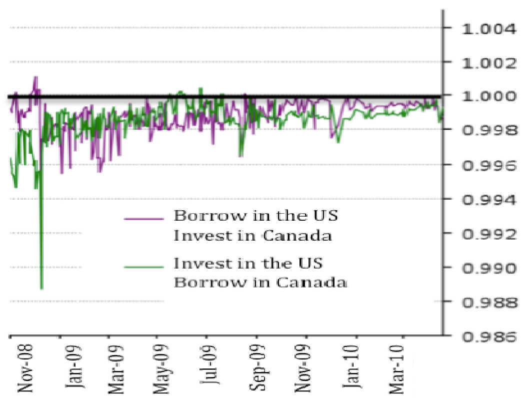

Fig. 6

From this exercise, it can be concluded that the interest rates used as reference to calculate arbitrage opportunities is essential in analyzing the results. When deciding in which asset to invest and in which asset to borrow, profits from arbitrage cannot be attributed if the composition of the assets is not similar.

7. Conclusions

In this paper, the foreign exchange arbitrage problem is analyzed using a circulation network approach. A network model was presented and the MCF algorithm was suggested to solve it. The model presented in this paper can be extended to consider a greater number of currencies and financial instruments.

In the examples analyzed, the possibility of circular arbitrage is observed. Circular arbitrage consists of simultaneous transactions in four markets: the spot exchange rate market, the forward exchange rate market, the domestic assets market and the foreign assets market. The arbitrageur examines the current quotes for the exchange rates and interest rates, and searches for the possibility of obtaining an instantaneous riskless gain using an appropriate combination of transactions in the markets. The speed at which the exchange and interest rates quotes change and the volume of transactions that take place in the exchange and asset markets are increasing over time; hence the relevance of this type of modeling. The circulation network approach facilitates the analysis of the arbitrage problem and helps to understand the transactions involved in a visual way.

The MCF algorithm obtains the transactions that an arbitrageur must perform to take advantage of the temporary gains that he can obtain by buying and selling currencies in the spot and forward markets, and by borrowing and lending in the domestic and international financial markets. It is important to note that this algorithm can solve any type of maximum flow problems and it can be applied to solve other types of financial problems such as management of cash flows, multinational payment systems or the development of investment portfolios.

As an extension to this paper, it would be of interest to include in the model the transaction costs involved in the exchange rate and financial markets operations. Moreover, another extension would be to model the subjective probability distributions over the future exchange rates to implement the speculative network model.

Market efficiency implies that there should not exist opportunities of obtaining profits from arbitrage. However, if there were no opportunities of arbitrage then the market would not have a mechanism through which it could reach the equilibrium. The study of exchange rate networks and the development of more efficient algorithms that search for deviations in the interest rate parity is essential in improving our understanding on the operation of markets.

References

Abeysekera, S. P., and Turtle, H. J. (1995). Long-Run Relations in Exchange Markets: A Test of Covered Interest Parity. *Journal of Financial Research, 18,* 431-47.

Ahuja, R. K., Magnanti, T. L., and Orlin, J. (1993). *Network Flows: Theory, Algorithms, and Applications.* Upper Saddle River: NJ, Prentice Hall.

Akram, Q. F., Rime, D., and Sarno, L. (2008). Arbitrage in the Foreign Exchange Market: Turning on the Microscope. *Journal of International Economics, 76,* 237-253.

Christofides, N., Hewins, R. D., and Salkin, G. R. (1979). Graph Theoretic Approaches to Foreign Exchange Operations. *Journal of Financial and Quantitative Analysis, 14,* 481-500.

Clinton, K. (1988). Transaction Costs and Covered Interest Arbitrage: Theory and Evidence. *Journal of Political Economy, 96,* 358-370.

Crum, R. L., and Nye, D. J. (1981). A Network Model of Insurance Company Cash Flow Management. *Mathematical Programming Study, 15,* 86-101.

Dantzig, G. B., and Infanger, G. (1993). Multi-Stage Stochastic Linear Programs for Portafolio Optimization. *Annuals of Operation Research, 45,* 59-76.

Deardorff, A. V. (1979). One-Way Arbitrage and its Implications for Foreign Exchange Markets. *Journal of Political Economy, 87,* 351-364.

Ellerman, D. P. (1984). Arbitrage Theory: A Mathematical Introduction. *SIAM Review, 26,* 241-261.

Ford, L. R., and Fulkerson, D. R. (1962). *Flows in Networks.* NJ: USA, Princeton University Press.

Frenkel, J. A., and Levich, R. M. (1975). Covered Interest Arbitrage: Unexploited Profits. *Journal of Political Economy, 83*, 325-338.

Ghosh, D. K. (1997). Profit Multiplier in Covered Currency Trading with Leverage. *Financial Review, 32,* 391-409.

Glover, F., and Jones, C. K. (1988). A Stochastic Generalized Network Model and Large-Scale Mean-Variance Algorithm for Portfolio Selection. *Journal of Information and Optimization Sciences, 9,* 137-152.

Goldberg, A. V., Plotkin, S. A., and Tardos, É. (1991). Combinatorial Algorithms for the Generalized Circulation Problem. *Mathematics of Operations Research, 16,* 351-381.

Goldfarb, D., Jin, O., and Orlin, J. B. (1997). Polynomial-Time Highest-Gain Augmenting Path Algorithms for the Generalized Circulation Problem. *Mathematics of Operations Research, 22,* 793-802.

Gondran, M., and Minoux, M. (1984). *Graphs and Algorithms.* NJ: USA, John Wiley & Sons Inc.

Hallwood, P. C., and MacDonald, R. (2000). *International Money and Finance*, Oxford: UK, Blackwell Publishers.

Jones, C. K. (2001). A Network Model for Foreign Exchange Arbitrage, Hedging and Speculation. *International Journal of Theoretical and Applied Finance, 4,* 837-852.

Radzik, T. (2004). Improving Time Bounds on Maximum Generalised Flow Computations by Contracting the Network. *Theoretical Computer Science, 312,* 75-97.

Rutenberg, D. P. (1970). Maneuvering Liquid Assets in a Multinational Company: Formulation and Deterministic Solution Procedures. *Management Science, 16,* 671-684.

Sercu, P., and Uppal, R. (1995). *International Financial Markets and the Firm*. Nashville: TN, South-Western College Publishing.

Shigeno, M. (2004). A Survey of Combinatorial Maximum Flow Algorithms on a Network with Gains. *Journal of the Operations Research Society of Japan, 47,* 244-264.

Srinivasan, V., and Kim, Y. H. (1986). Payments Netting in International Cash Management: A Network Optimization Approach*. Journal of International Business Studies, 17,* 1-20.

Onaga, K. (1966). Dynamic Programming of Optimum Flows in Lossy Communication Nets. *IEEE Transactions on Circuit Theory, 13,* 282-287.

Onaga, K. (1967). Optimal Flows in General Communication Networks. *Journal of the Franklin Institute, 283,* 308-327.

Orlin, J. B. (1993). A Faster Strongly Polynomial Minimum Cost Flow Algorithm. *Operations Research, 41,* 338-350.

Tiernan, J. C. (1970). An Efficient Search Algorithm to Find the Elementary Circuits of a Graph. *Communnications of the ACM, 13,* 722-726.

Truemper, K. (1977). On Max Flows with Gains and Pure Min-Cost Flows, SIAM. *Journal on Applied Mathematics, 32,* 450-456.